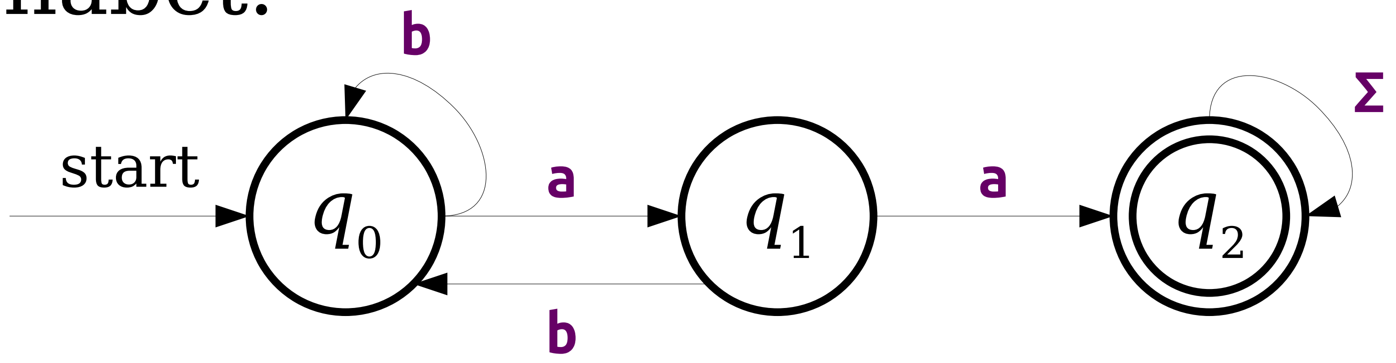
Finite Automata

Part Three

Recap from Last TIme

DFAs

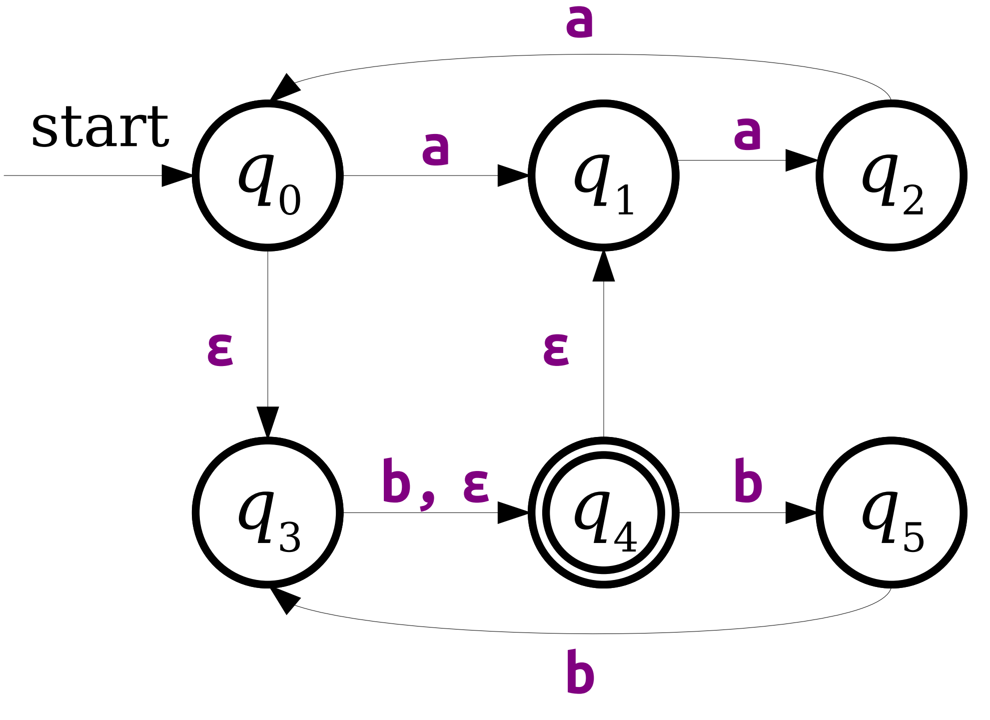
- A ***DFA*** is a
  - ***D***eterministic
  - ***F***inite
  - ***A***utomaton
- Every state must have exactly one transition defined for each symbol in the alphabet.



If D is a DFA, the ***language of D***, denoted $\mathscr{L}(D)$, is $\{\ w \in \Sigma^* \mid D \text{ accepts } w\ \}$.

NFAs

- An ***NFA*** is a
  - ***N***ondeterministic
  - ***F***inite
  - ***A***utomaton
- NFAs have no restrictions on how many transitions are allowed per state.
- They can also use ε-transitions.
- An NFA accepts a string *w* if there is some sequence of choices that leads to an accepting state.



Massive Parallelism

- An NFA can be thought of as a DFA that can be in many states at once.
- At each point in time, when the NFA needs to follow a transition, it tries all the options at the same time.
- The NFA accepts if *any* of the states that are active at the end are accepting states. It rejects otherwise.

New Stuff!

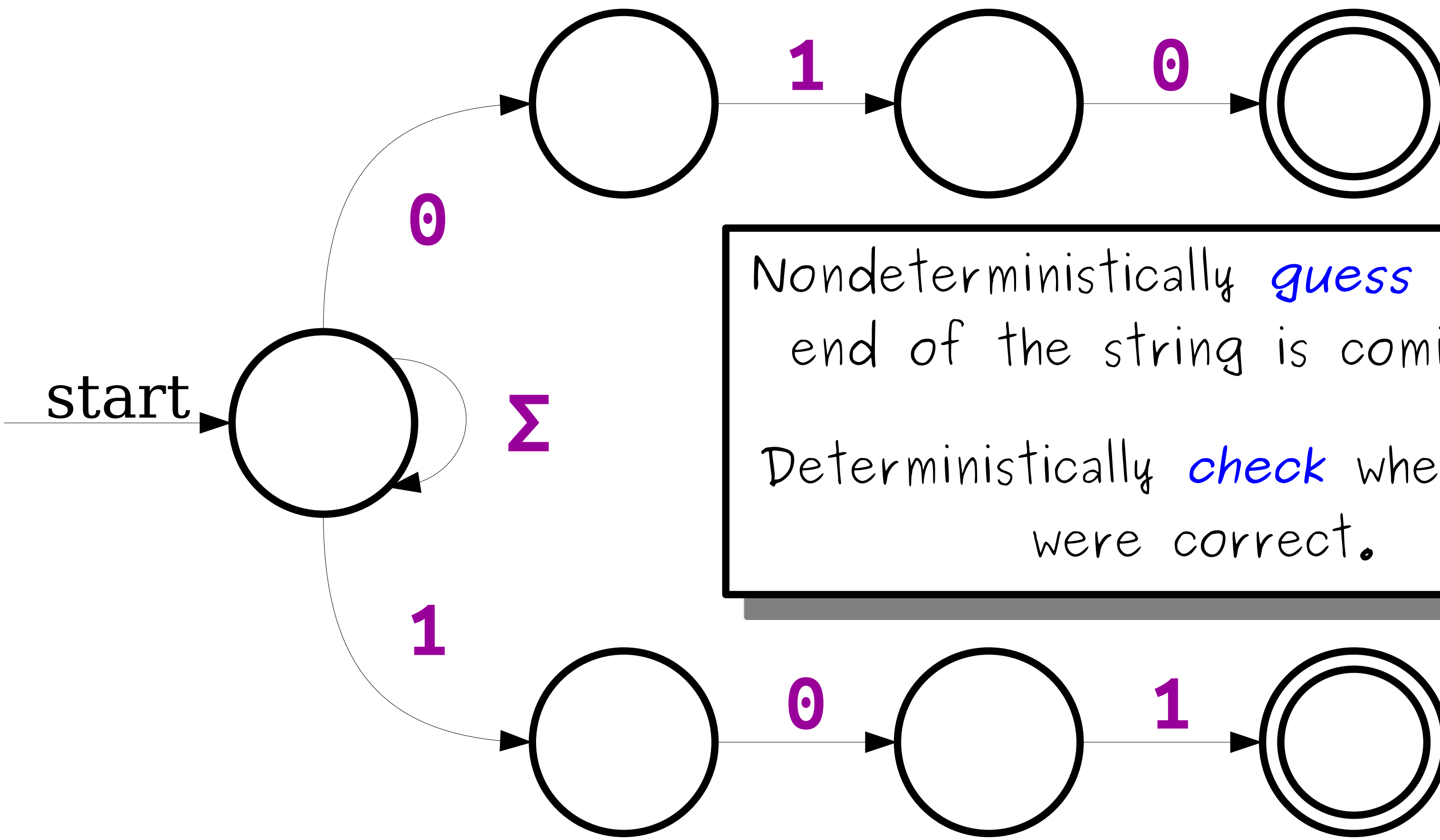
Designing NFAs

Designing NFAs

- ***Embrace the nondeterminism!***
- Good model: ***Guess-and-check***:
  - Is there some information that you'd really like to have? Have the machine *nondeterministically guess* that information.
  - Then, have the machine *deterministically check* that the choice was correct.
- The *guess* phase corresponds to trying lots of different options.
- The *check* phase corresponds to filtering out bad guesses or wrong options.

Guess-and-Check

$$L = \{\, w \in \{\mathtt{0}, \mathtt{1}\}^* \mid w \text{ ends in } \mathtt{010} \text{ or } \mathtt{101} \,\}$$

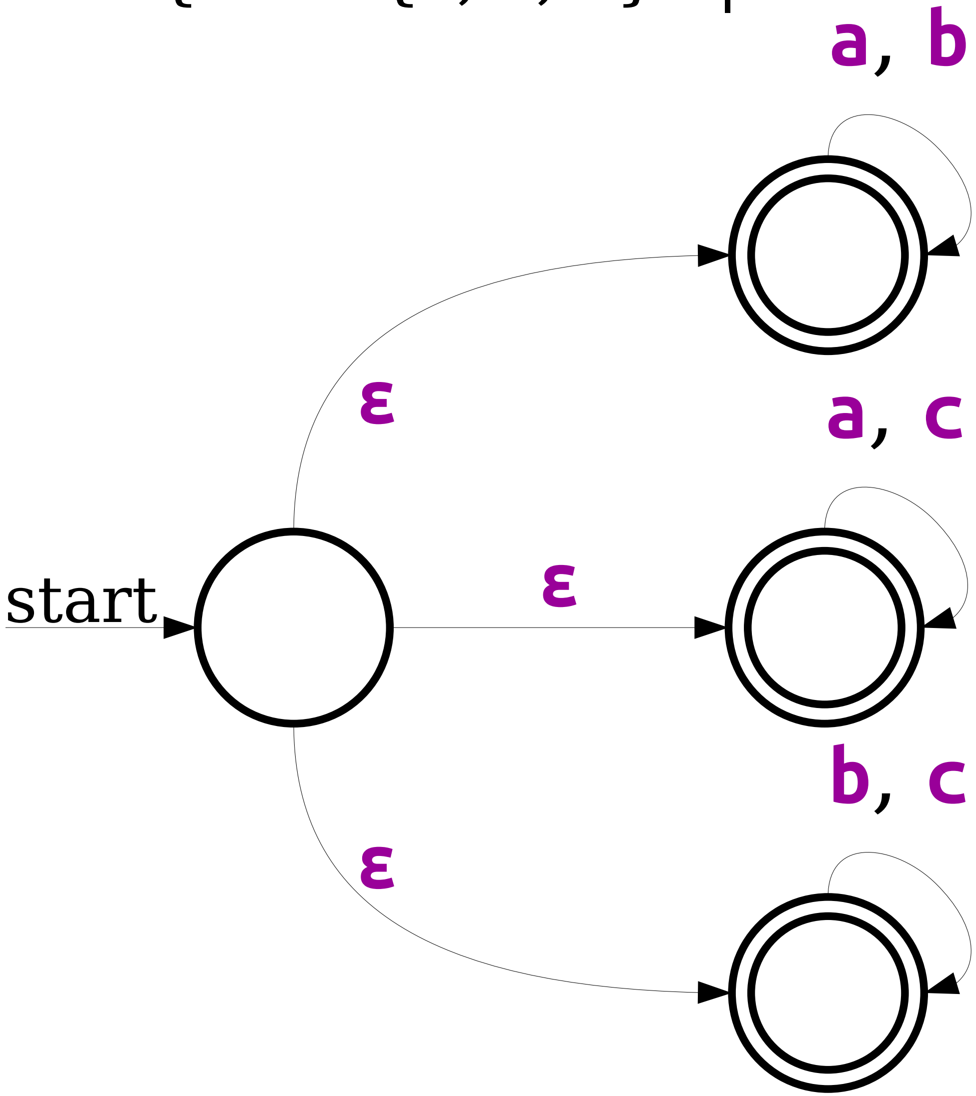


Nondeterministically *guess* when the end of the string is coming up.

Deterministically *check* whether you were correct.

Guess-and-Check

$L = \{ \ w \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^* \mid \text{at least one of } \mathtt{a}, \mathtt{b}, \text{ or } \mathtt{c} \text{ is not in } w \ \}$

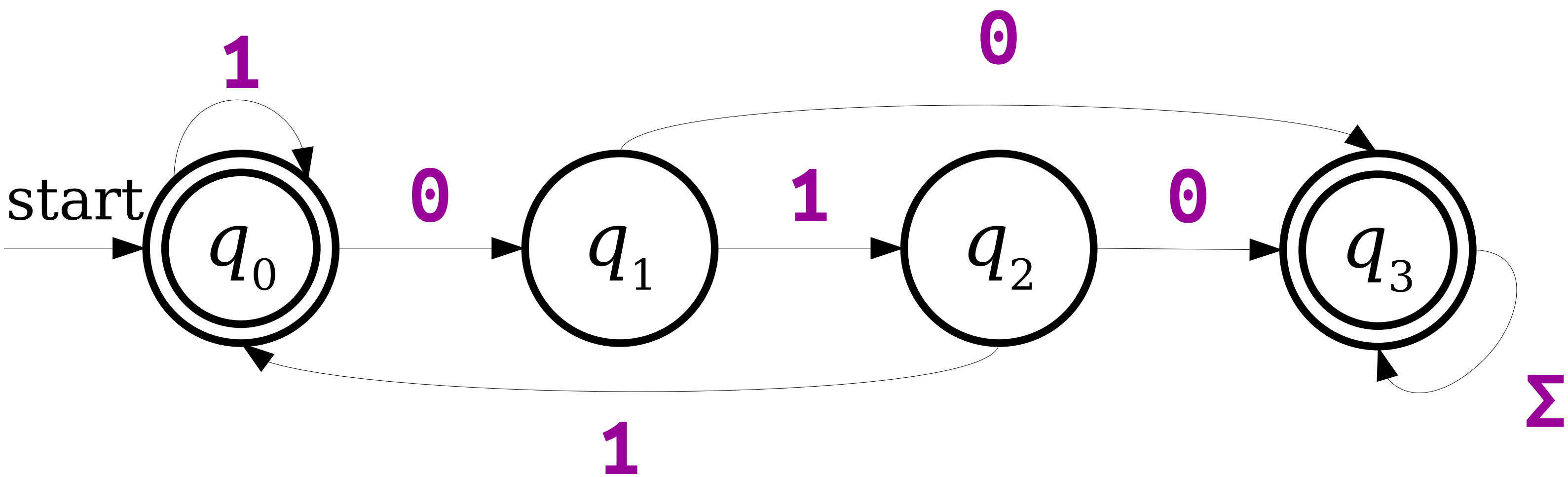


Nondeterministically *guess* which character is missing.

Deterministically *check* whether that character is indeed missing.

Implementing DFAs and NFAs

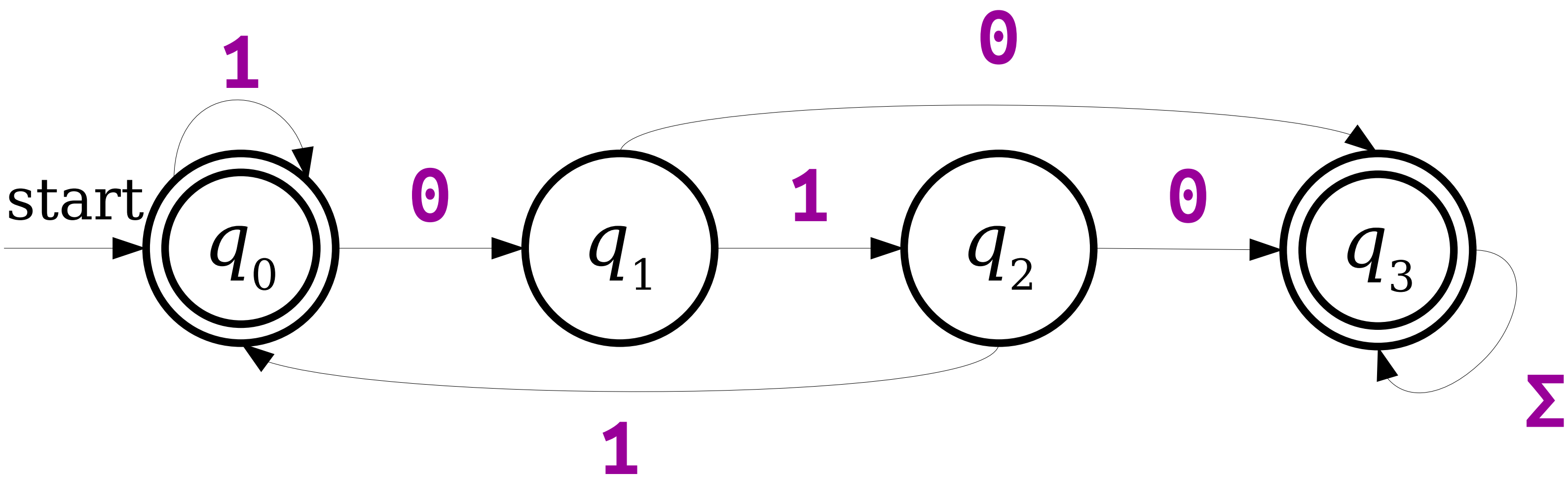
Tabular DFAs



	0	1
$*q_0$	q_1	q_0
q_1	q_3	q_2
q_2	q_3	q_0
$*q_3$	q_3	q_3

These stars indicate accepting states.

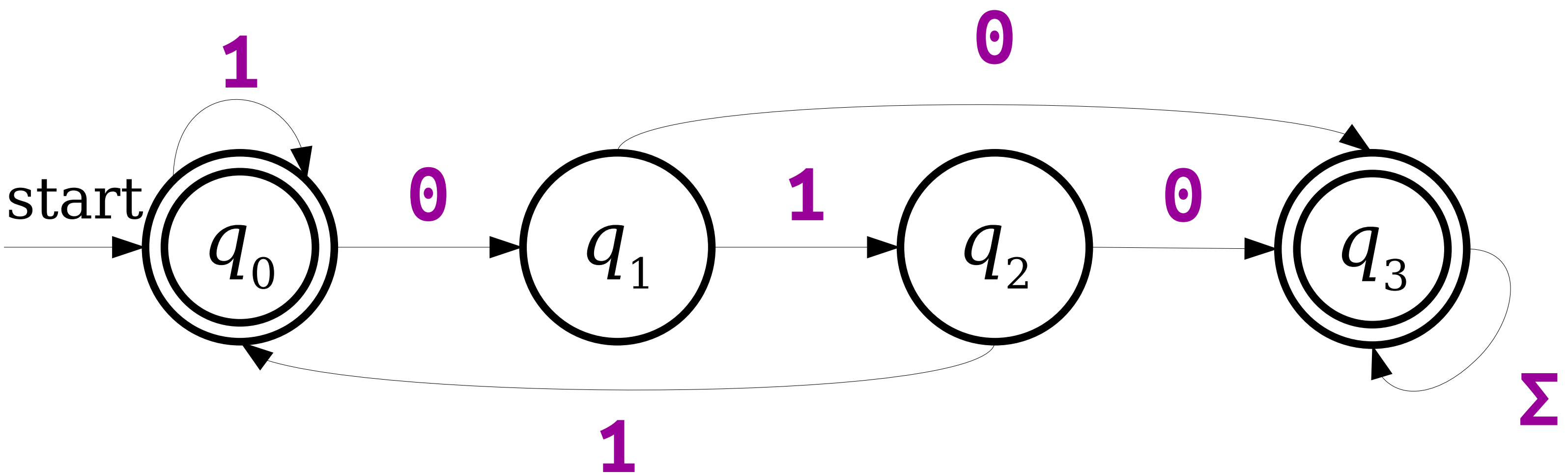
Tabular DFAs



	0	1
$*q_0$	q_1	q_0
q_1	q_3	q_2
q_2	q_3	q_0
$*q_3$	q_3	q_3

Since this is the first row, it's the start state.

Tabular DFAs



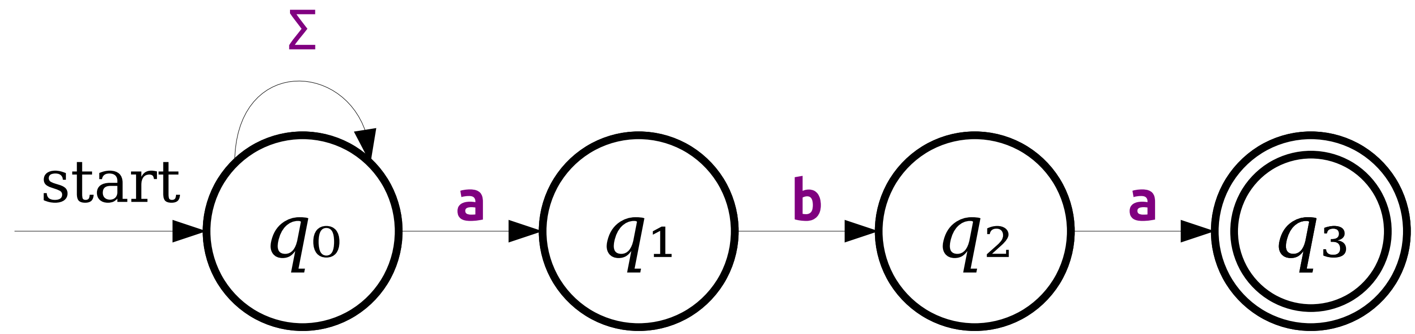
	0	1
$*q_0$	q_1	q_0
q_1	q_3	q_2
q_2	q_3	q_0
$*q_3$	q_3	q_3

Question to ponder: Why isn't there a column here for Σ?

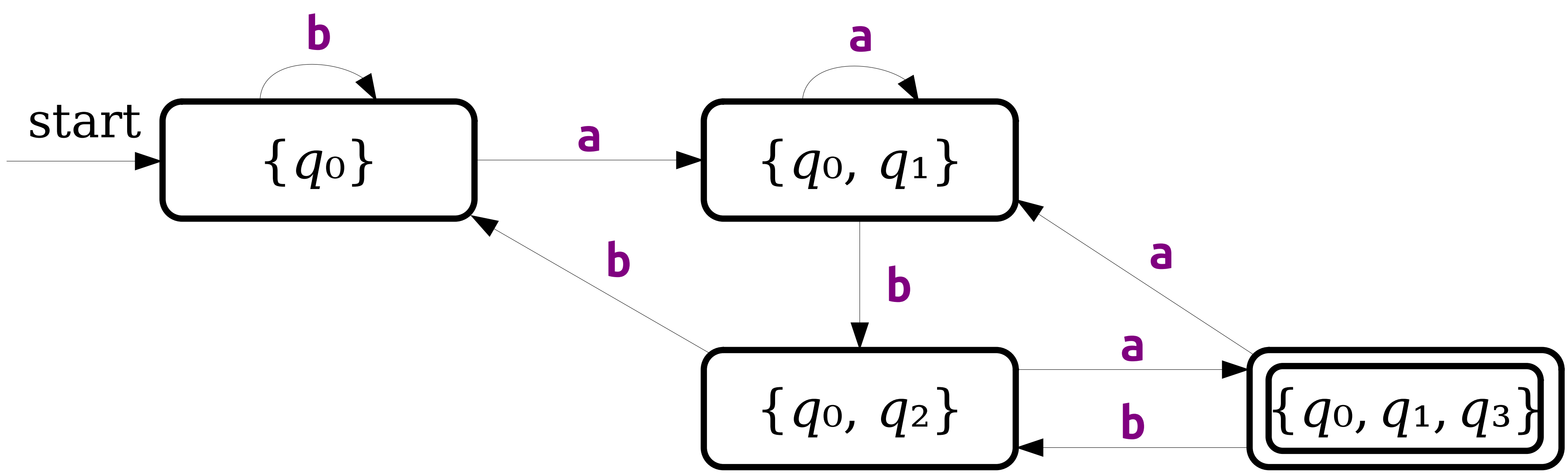
Code? In a Theory Class?

```
int kTransitionTable[kNumStates][kNumSymbols] = {
     {0, 0, 1, 3, 7, 1, …},
      …
};
bool kAcceptTable[kNumStates] = {
    false,
    true,
    true,
    …
};
bool SimulateDFA(string input) {
    int state = 0;
    for (char ch: input) {
        state = kTransitionTable[state][ch];
    }
    return kAcceptTable[state];
}
```

Thought Experiment:
How would you simulate an NFA in software?



	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$
$\{q_0, q_2\}$	$\{q_0, q_1, q_3\}$	$\{q_0\}$
$*\{q_0, q_1, q_3\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$



The Subset Construction

- This procedure for turning an NFA for a language L into a DFA for a language L is called the ***subset construction***.
  - It's sometimes called the ***powerset construction***; it's different names for the same thing!
- Intuitively:
  - Each state in the DFA corresponds to a set of states from the NFA.
  - Each transition in the DFA corresponds to what transitions would be taken in the NFA when using the massive parallel intuition.
  - The accepting states in the DFA correspond to which sets of states would be considered accepting in the NFA when using the massive parallel intuition.
- There's an online ***Guide to the Subset Construction*** with a more elaborate example involving ε-transitions and cases where the NFA dies; check that for more details.

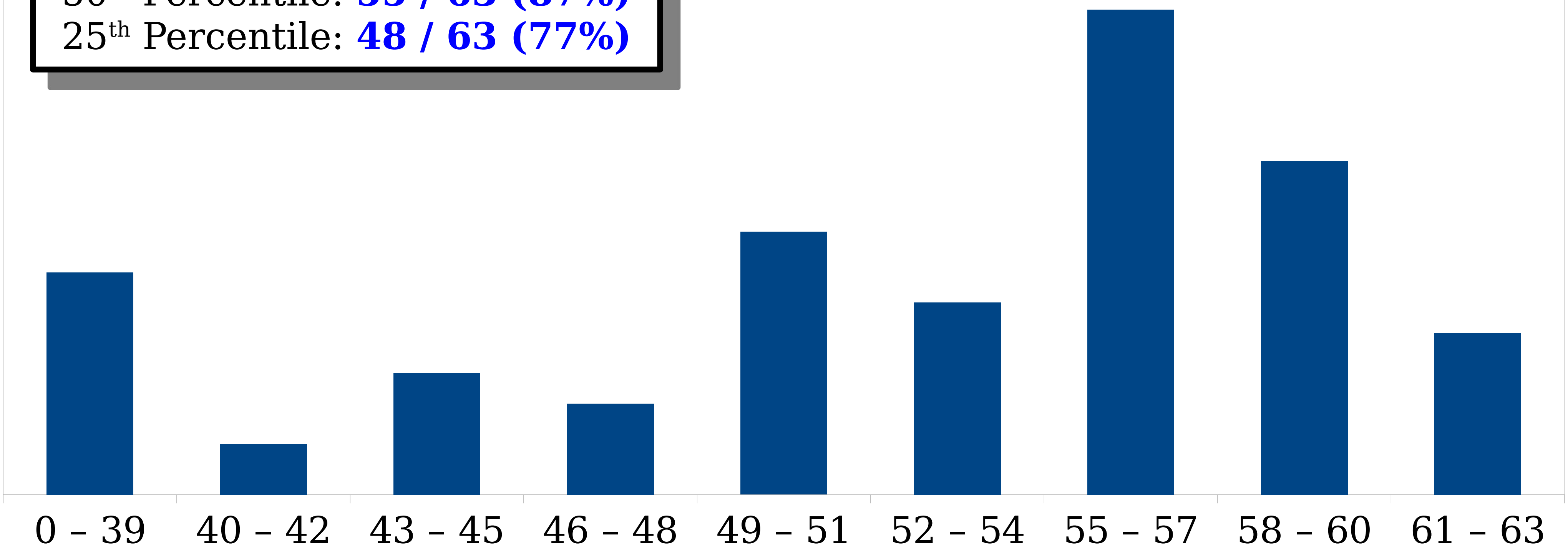
The Subset Construction

- In converting an NFA to a DFA, the DFA's states correspond to sets of NFA states.
- ***Useful fact:*** $|\wp(S)| = 2^{|S|}$ for any finite set S.
- In the worst-case, the construction can result in a DFA that is *exponentially larger* than the original NFA.
- ***Question to ponder:*** Can you find a family of languages that have NFAs of size n, but no DFAs of size less than 2^n?

Time-Out for Announcements!

Problem Set Four Grades

75th Percentile: **58 / 63 (92%)**
50th Percentile: **55 / 63 (87%)**
25th Percentile: **48 / 63 (77%)**



Many of these grades are because folks forgot to list partners – please check to make sure you're getting credit for the work you're doing, and let us know if your partner forgot to add you.

Problem Set Six

- Problem Set Five was due at 2:30PM today.
- Problem Set Six goes out today. It’s due next Friday at 2:30PM.
  - Design DFAs and NFAs for a range of problems!
  - Explore formal language theory!
  - See some clever applications!

Midterm Revise-and-Resubmit

- The first midterm exam is now open for revise-and-resubmit.
- For each problem you'd like to submit a new answer to, file a regrade request and type in your new answer.
- The deadline to submit is ***Monday at 2:30PM***.

Extra Practice Problems 2

- Looking for more practice with functions, graphs, and induction? We’ve just posted Extra Practice Problems 2 to the course website.
- It’s a collection of 30 problems on these topics, plus others from PS3 – PS5.
- Solutions are available as well.

~~Your Questions~~

We'll do this next time when we have a little more breathing room.

Back to CS103!

The Regular Languages

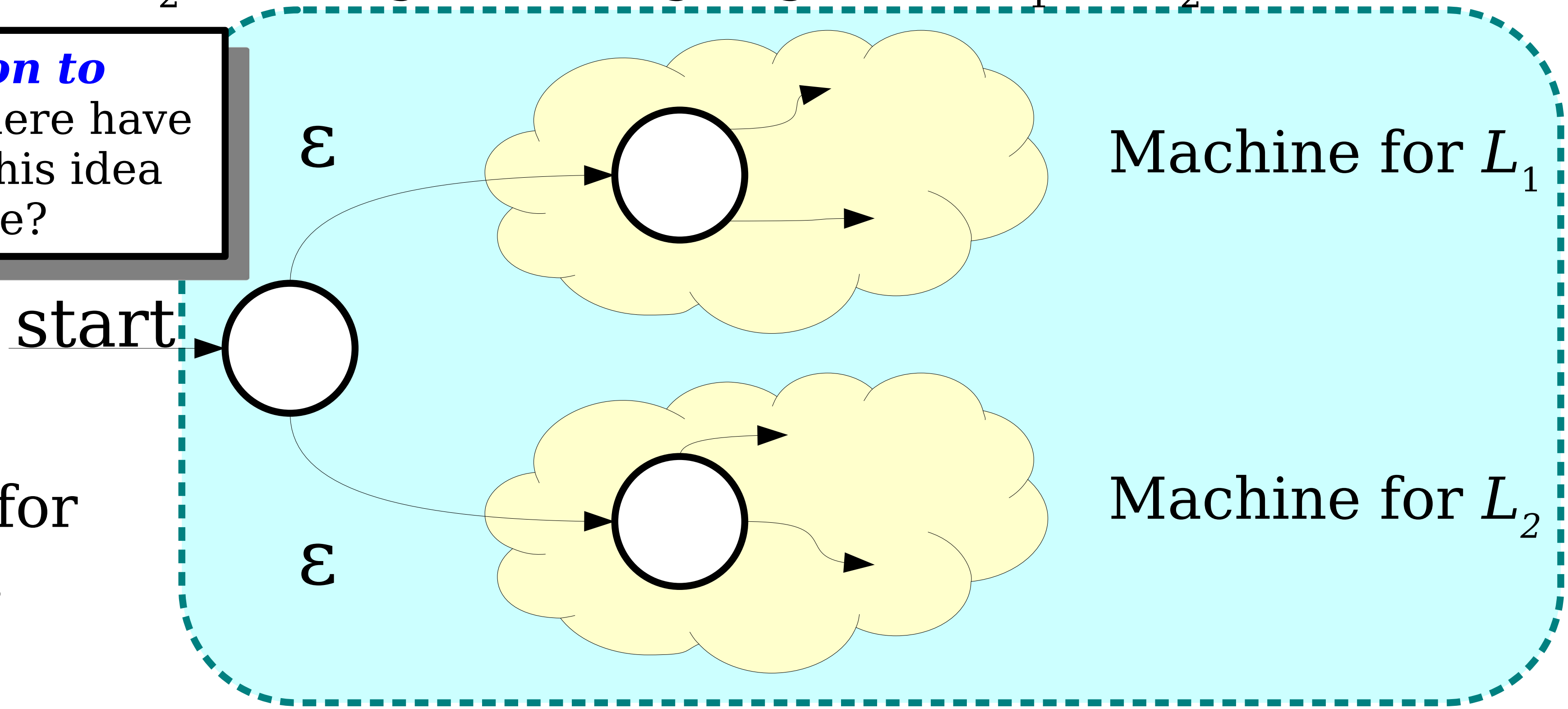
Regular Languages

- Let $L \subseteq \Sigma^*$ be a language.
- We say that L is a ***regular language*** if there is a DFA D where $\mathscr{L}(D) = L$.
- Equivalently, L is a regular language if there is an NFA N where $\mathscr{L}(N) = L$.
- Key questions:
  - What do the regular languages "feel" like?
  - What properties do they have?
  - What languages *aren't* regular?

Closure Under Union

- If L_1 and L_2 are languages over the alphabet Σ, the language $L_1 \cup L_2$ is the language of all strings in at least one of the two languages.
- If L_1 and L_2 are regular languages, is $L_1 \cup L_2$?

Question to ponder: where have you seen this idea before?



Machine for $L_1 \cup L_2$

Concatenation

String Concatenation

- If $w \in \Sigma^*$ and $x \in \Sigma^*$, the ***concatenation*** of w and x, denoted ***wx***, is the string formed by tacking all the characters of x onto the end of w.
- Example: if w = **quo** and x = **kka**, the concatenation wx = **quokka**.
- This is analogous to the + operator for strings in many programming languages.
- Some facts about concatenation:
  - The empty string ε is the ***identity element*** for concatenation:

$$w\varepsilon = \varepsilon w = w$$

  - Concatenation is ***associative***:

$$wxy = w(xy) = (wx)y$$

Concatenation

- The ***concatenation*** of two languages L_1 and L_2 over the alphabet Σ is the language

  $$\boldsymbol{L_1 L_2 = \{ x \mid \exists w_1 \in L_1.\ \exists w_2 \in L_2.\ x = w_1 w_2 \}}$$

Concatenation Example

- Let Σ = { **a**, **b**, …, **z**, **A**, **B**, …, **Z** } and consider these languages over Σ:
  - ***Noun*** = { **Puppy**, **Rainbow**, **Whale**, … }
  - ***Verb*** = { **Hugs**, **Juggles**, **Loves**, … }
  - ***The*** = { **The** }
- The language ***TheNounVerbTheNoun*** is
  - { **ThePuppyHugsTheWhale**, **TheWhaleLovesTheRainbow**, **TheRainbowJugglesTheRainbow**, … }

Concatenation

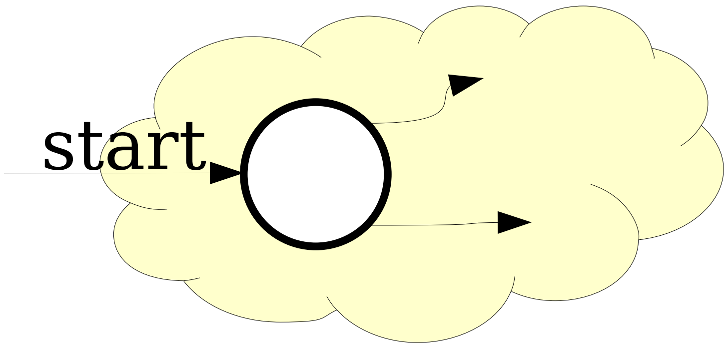
- The ***concatenation*** of two languages L_1 and L_2 over the alphabet Σ is the language

$$L_1L_2 = \{ x \mid \exists w_1 \in L_1.\ \exists w_2 \in L_2.\ x = w_1w_2 \}$$

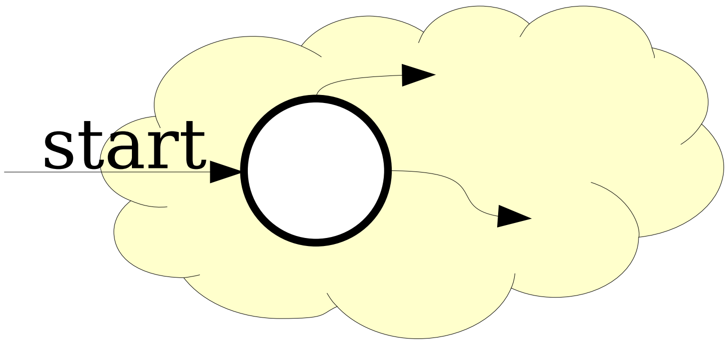
- Two views of L_1L_2:
  - The set of all strings that can be made by concatenating a string in L_1 with a string in L_2.
  - The set of strings that can be split into two pieces: a piece from L_1 and a piece from L_2.

Closure Under Concatenation

- If L_1 and L_2 are regular languages, is L_1L_2?
- Intuition – can we split a string w into two strings xy such that $x \in L_1$ and $y \in L_2$?



Machine for L_1



Machine for L_2

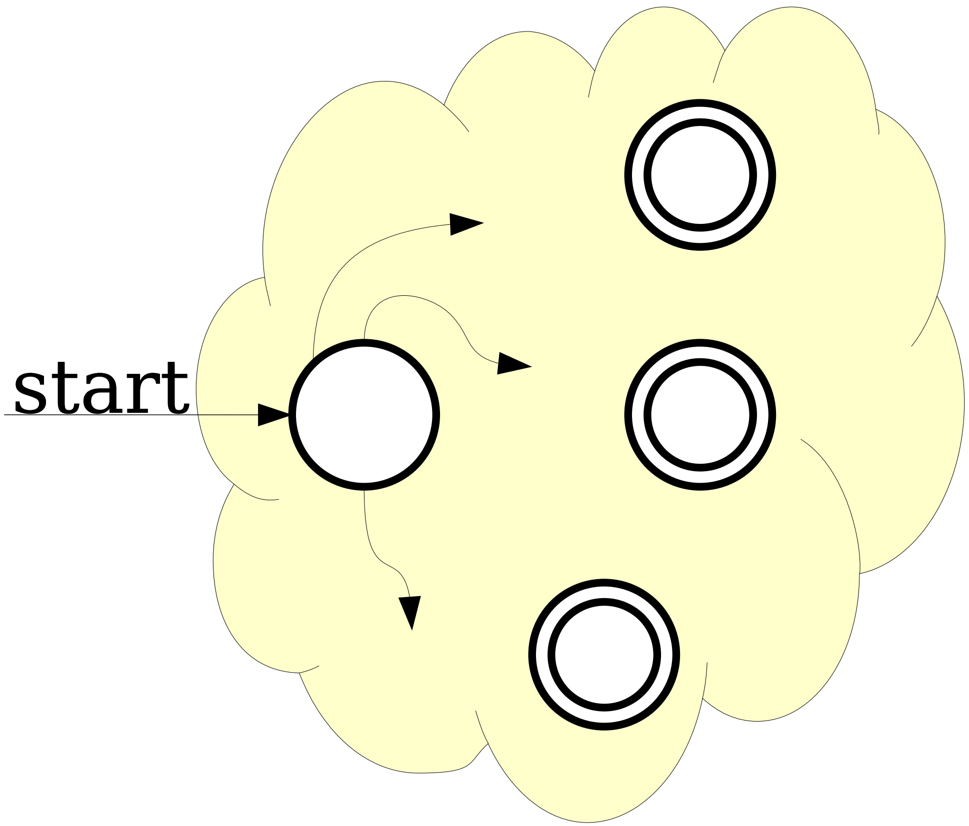
b	**o**	**o**	**k**

k	**e**	**e**	**p**	**e**	**r**

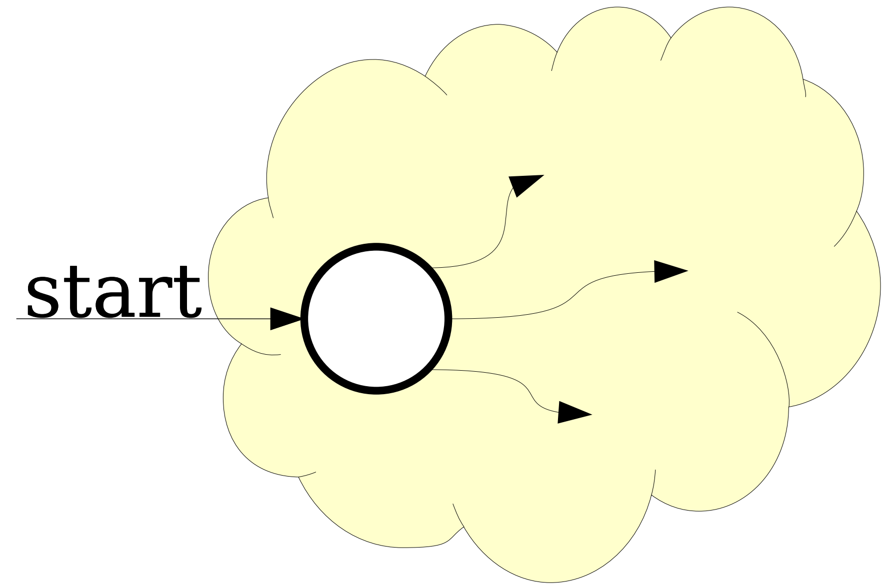
Closure Under Concatenation

- If L_1 and L_2 are regular languages, is L_1L_2?
- Intuition – can we split a string *w* into two strings *xy* such that $x \in L_1$ and $y \in L_2$?
- ***Idea***:
  - Run a DFA/NFA for L_1 on *w*.
  - Whenever it reaches an accepting state, optionally hand the rest of *w* to a DFA/NFA for L_2.
  - If the automaton for L_2 accepts the rest, $w \in L_1L_2$.
  - If the automaton for L_2 rejects the remainder, the split was incorrect.

Closure Under Concatenation

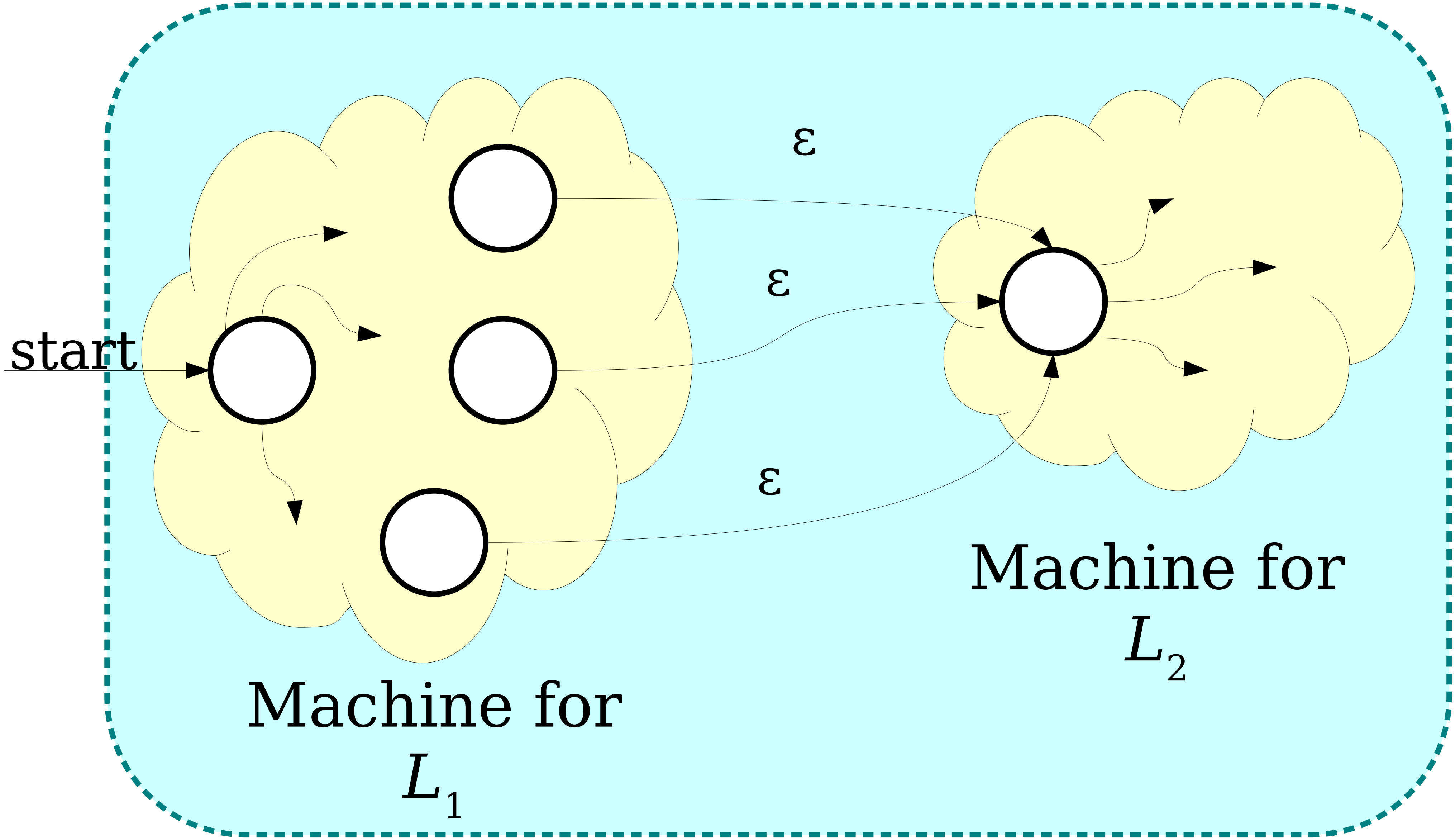


Machine for L_1



Machine for L_2

Closure Under Concatenation



Machine for L_1L_2

The Kleene Star

Lots and Lots of Concatenation

- Consider the language L = { aa, b }
- LL is the set of strings formed by concatenating pairs of strings in L.

  { aaaa, aab, baa, bb }

- LLL is the set of strings formed by concatenating triples of strings in L.

  { aaaaaa, aaaab, aabaa, aabb, baaaa, baab, bbaa, bbb}

- $LLLL$ is the set of strings formed by concatenating quadruples of strings in L.

  { aaaaaaaa, aaaaaab, aaaabaa, aaaabb, aabaaaa, aabaab, aabbaa, aabbb, baaaaaa, baaaab, baabaa, baabb, bbaaaa, bbaab, bbbaa, bbbb}

Language Exponentiation

- We can define what it means to “exponentiate” a language as follows:
- $L^0 = \{\varepsilon\}$
  - Intuition: The only string you can form by gluing no strings together is the empty string.
  - Notice that $\{\varepsilon\} \neq \emptyset$. Can you explain why?
- $L^{n+1} = LL^n$
  - Idea: Concatenating (n+1) strings together works by concatenating n strings, then concatenating one more.
- ***Question to ponder:*** Why define $L^0 = \{\varepsilon\}$?
- ***Question to ponder:*** What is $\emptyset^0$?

The Kleene Star

The Kleene Closure

- An important operation on languages is the ***Kleene Closure***, which is defined as

$$\boldsymbol{L^* = \{ w \in \Sigma^* \mid \exists n \in \mathbb{N}.\ w \in L^n \}}$$

- Mathematically:

$$\boldsymbol{w \in L^* \quad \leftrightarrow \quad \exists n \in \mathbb{N}.\ w \in L^n}$$

- Intuitively, L^* is the language all possible ways of concatenating zero or more strings in L together, possibly with repetition.
- ***Question to ponder:*** What is $\varnothing^*$?

The Kleene Closure

If L = { **a**, **bb** }, then L^* = {

ε,

a, **bb**,

aa, **abb**, **bba**, **bbbb**,

aaa, **aabb**, **abba**, **abbbb**, **bbaa**, **bbabb**, **bbbba**, **bbbbbb**,

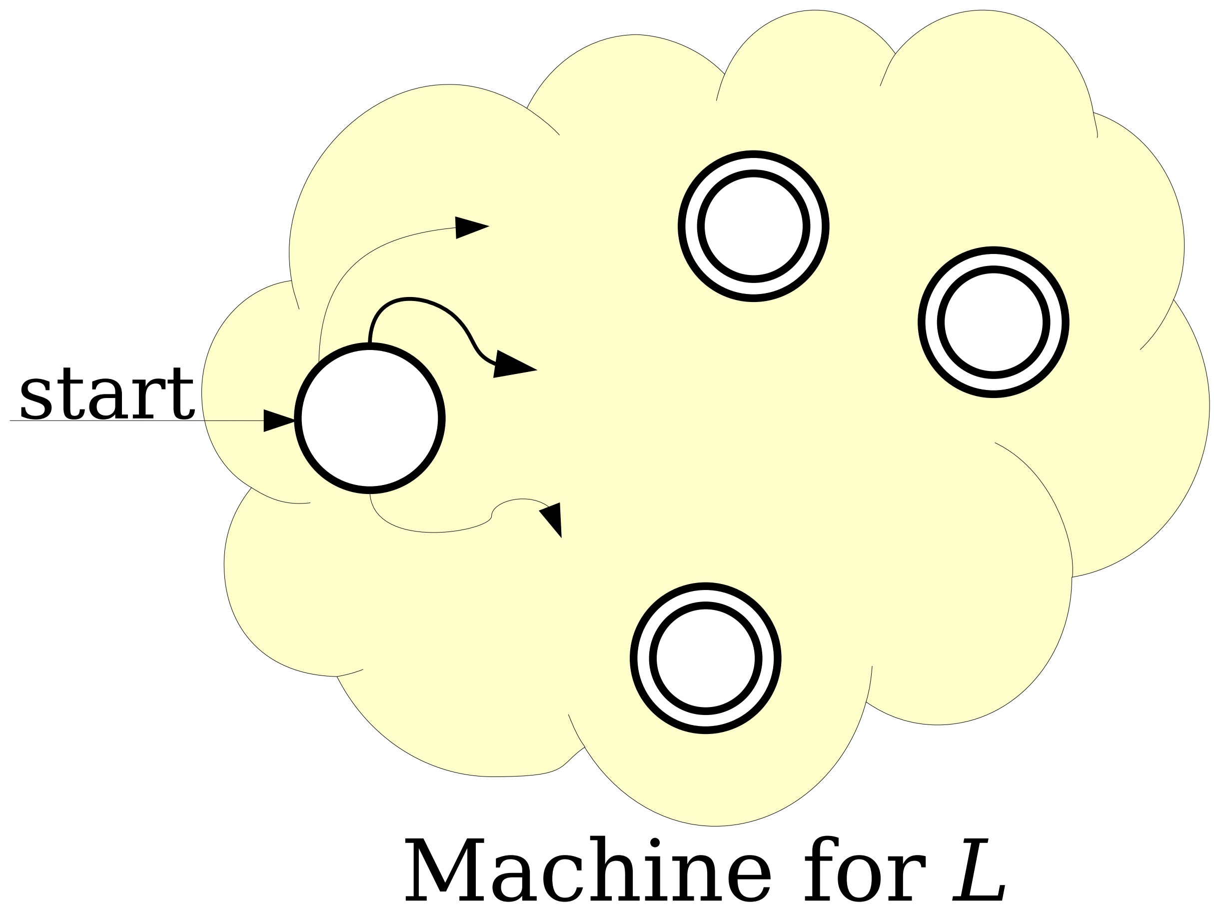
…

}

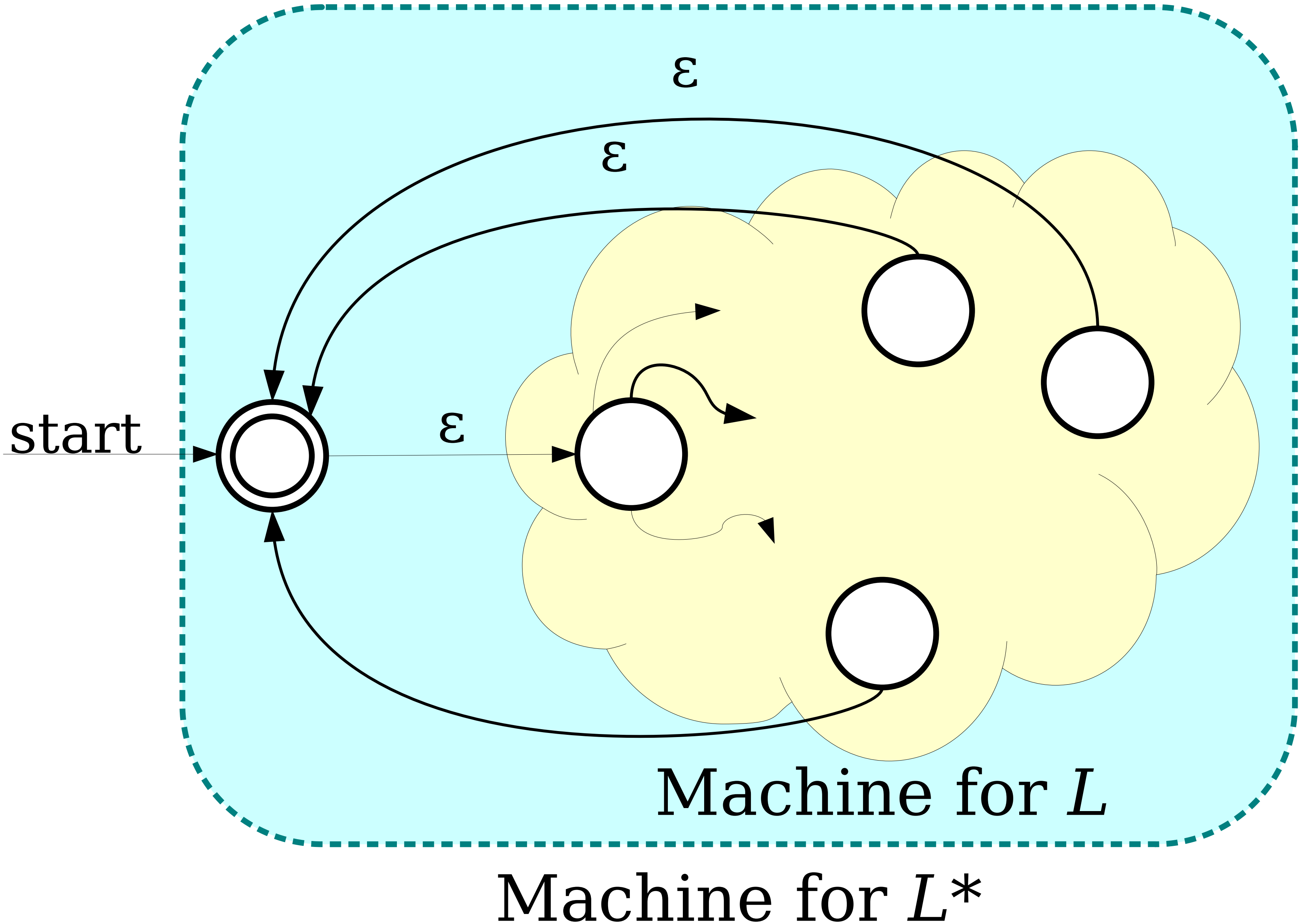
Think of L* as the set of strings you can make if you have a collection of stamps – one for each string in L – and you form every possible string that can be made from those stamps.

Idea: Can we convert an NFA for a language L to an NFA for language L^*?

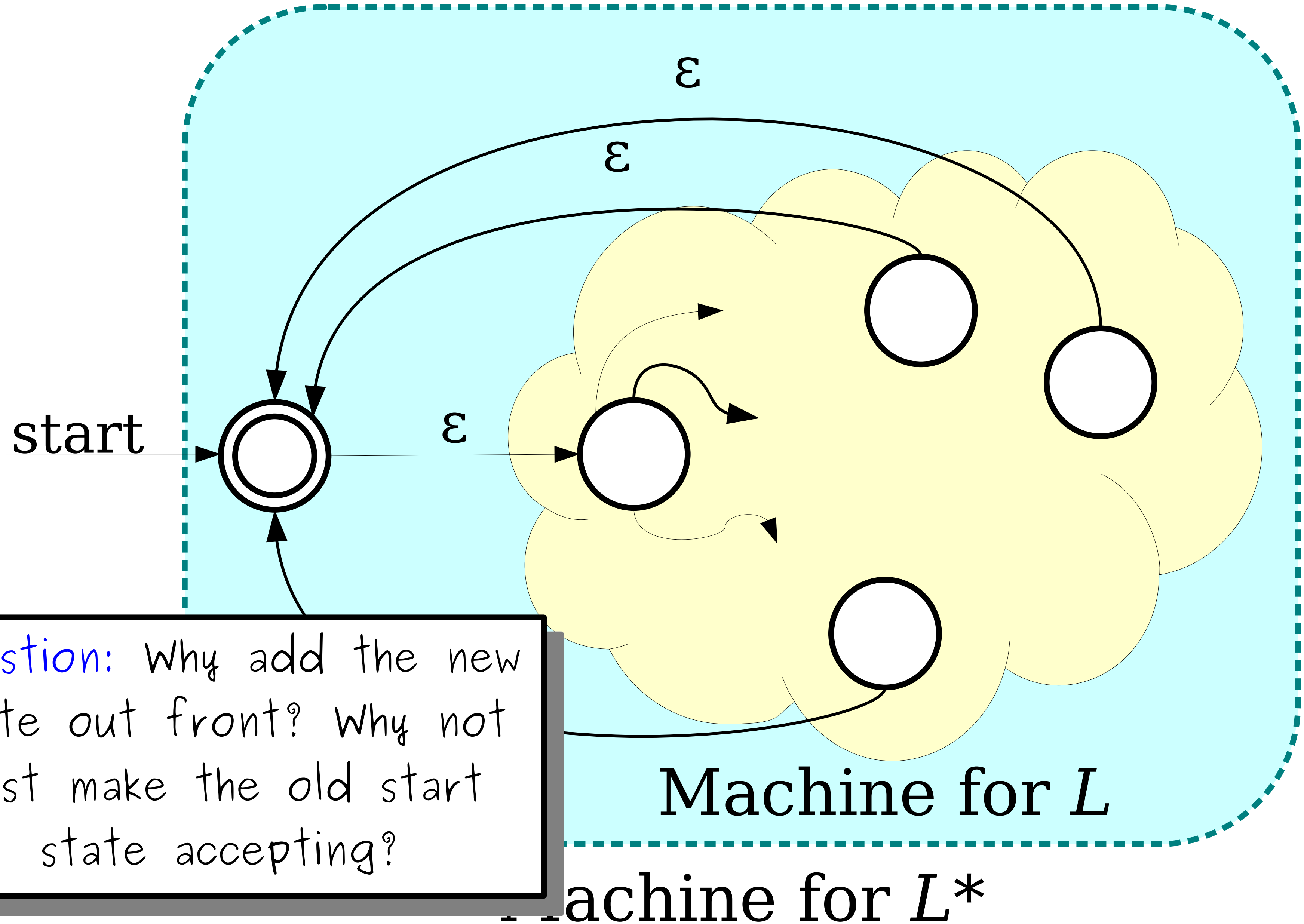
The Kleene Star



The Kleene Star



The Kleene Star



Question: Why add the new state out front? Why not just make the old start state accepting?

Closure Properties

- ***Theorem:*** If L_1 and L_2 are regular languages over an alphabet Σ, then so are the following languages:
  - $L_1 \cup L_2$
  - $L_1 L_2$
  - L_1^*
- These properties are called ***closure properties of the regular languages***.

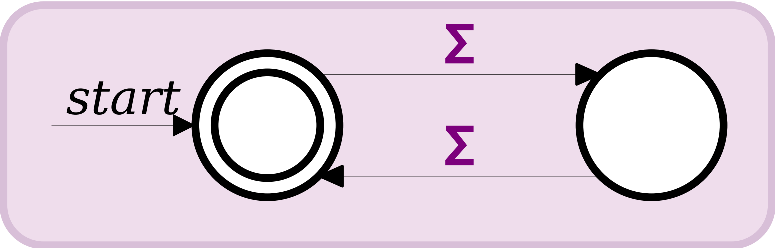
In the Appendix

- The appendix to this slide deck contains
  - three examples of applying closure properties to concrete languages and how it relates to machines, and
  - two more closure properties beyond what we saw today.
- Take a look – there's some really cool stuff here!

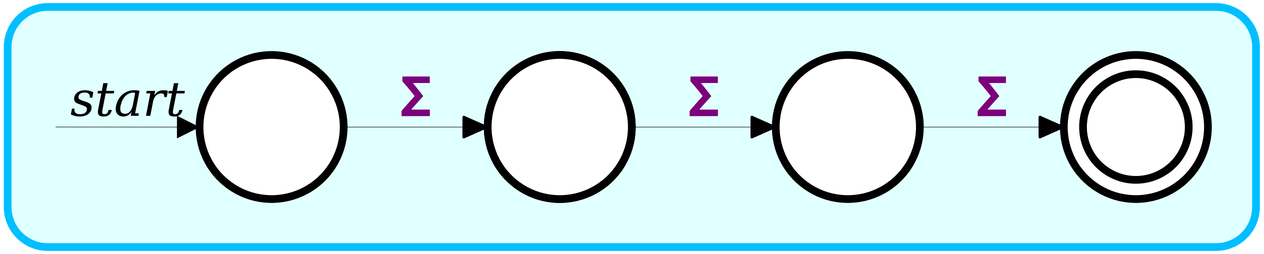
Next Time

- ***Regular Expressions***
  - Building languages from the ground up!
- ***Thompson's Algorithm***
  - A UNIX Programmer in Theoryland.
- ***Kleene's Theorem***
  - From machines to programs!

Appendix 1: Closure Properties Applied



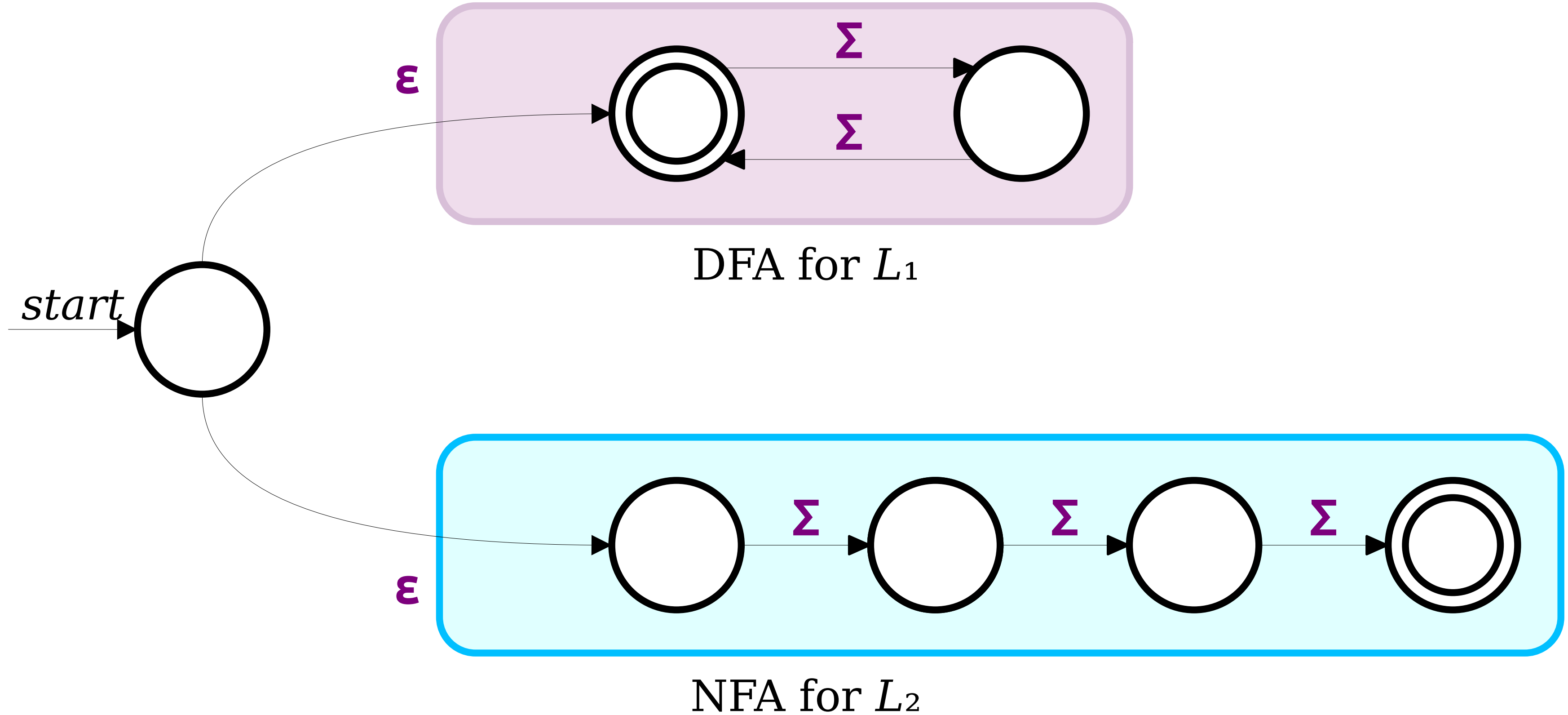
DFA for L_1



NFA for L_2

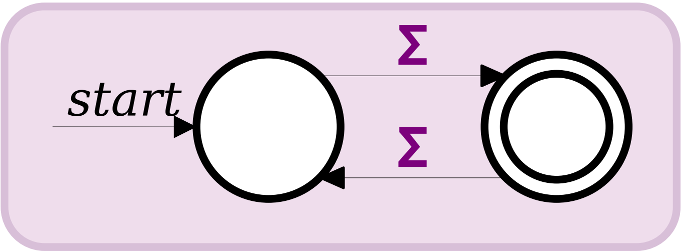
$$L_1 = \{\ w \in \{\texttt{a}, \texttt{b}\}^* \mid w \text{ has even length }\}$$
$$L_2 = \{\ w \in \{\texttt{a}, \texttt{b}\}^* \mid w \text{ has length exactly three }\}$$

Construct an NFA for $L_1 \cup L_2$.

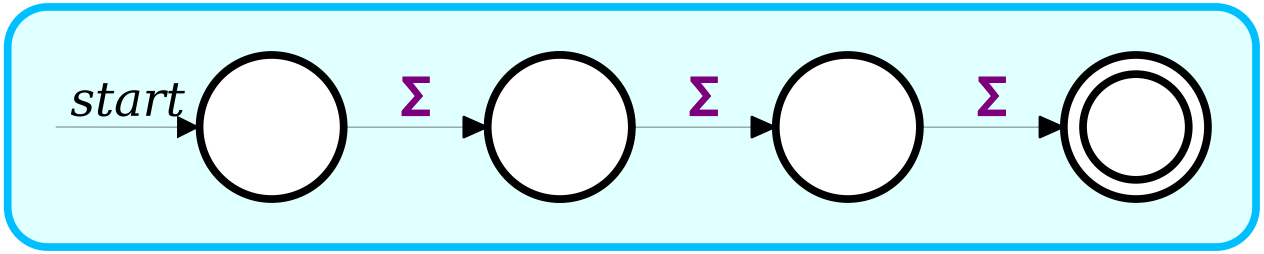


$L_1 = \{ w \in \{\mathtt{a}, \mathtt{b}\}^* \mid w$ has even length $\}$
$L_2 = \{ w \in \{\mathtt{a}, \mathtt{b}\}^* \mid w$ has length exactly three $\}$

Construct an NFA for $L_1 \cup L_2$.



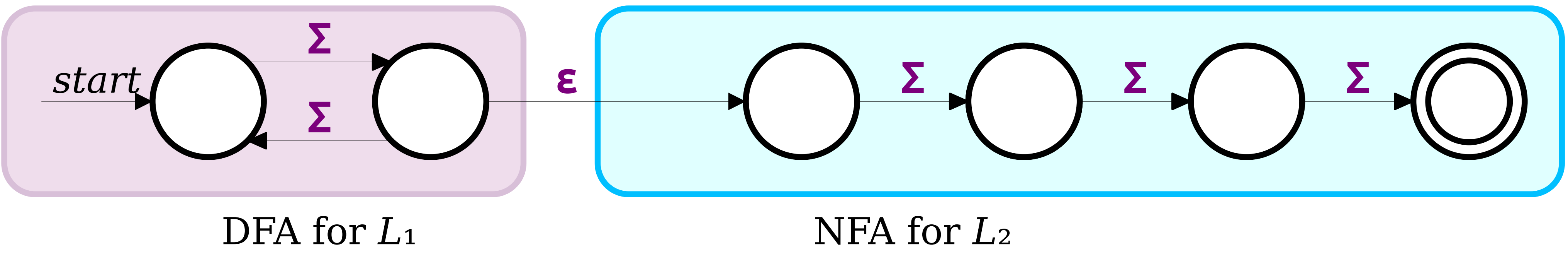
DFA for L_1



NFA for L_2

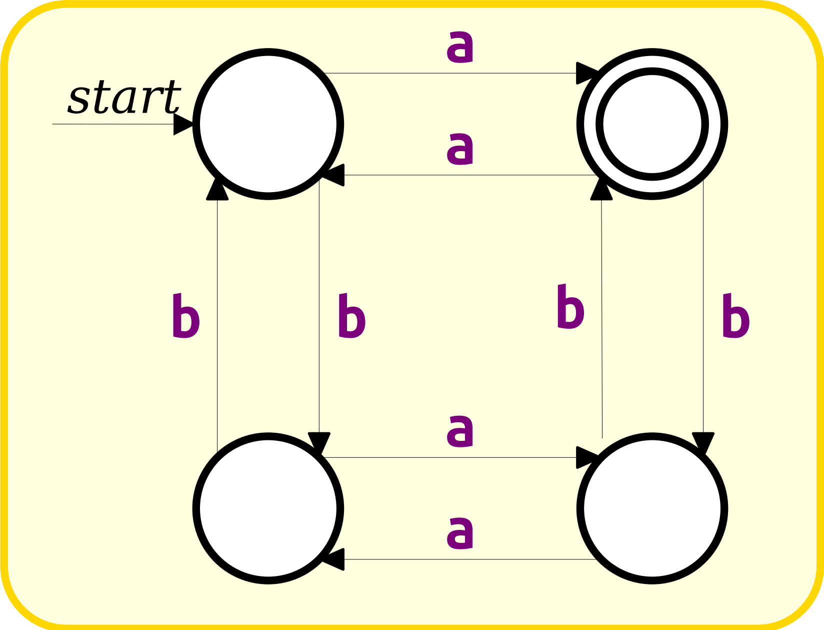
$L_1 = \{ w \in \{\mathtt{a}, \mathtt{b}\}^* \mid w \text{ has } \textit{odd} \text{ length } \}$
$L_2 = \{ w \in \{\mathtt{a}, \mathtt{b}\}^* \mid w \text{ has length exactly three } \}$

Construct an NFA for $L_1 L_2$.



$L_1 = \{\ w \in \{\texttt{a}, \texttt{b}\}^* \mid w \text{ has } \textit{odd} \text{ length}\ \}$
$L_2 = \{\ w \in \{\texttt{a}, \texttt{b}\}^* \mid w \text{ has length exactly three}\ \}$

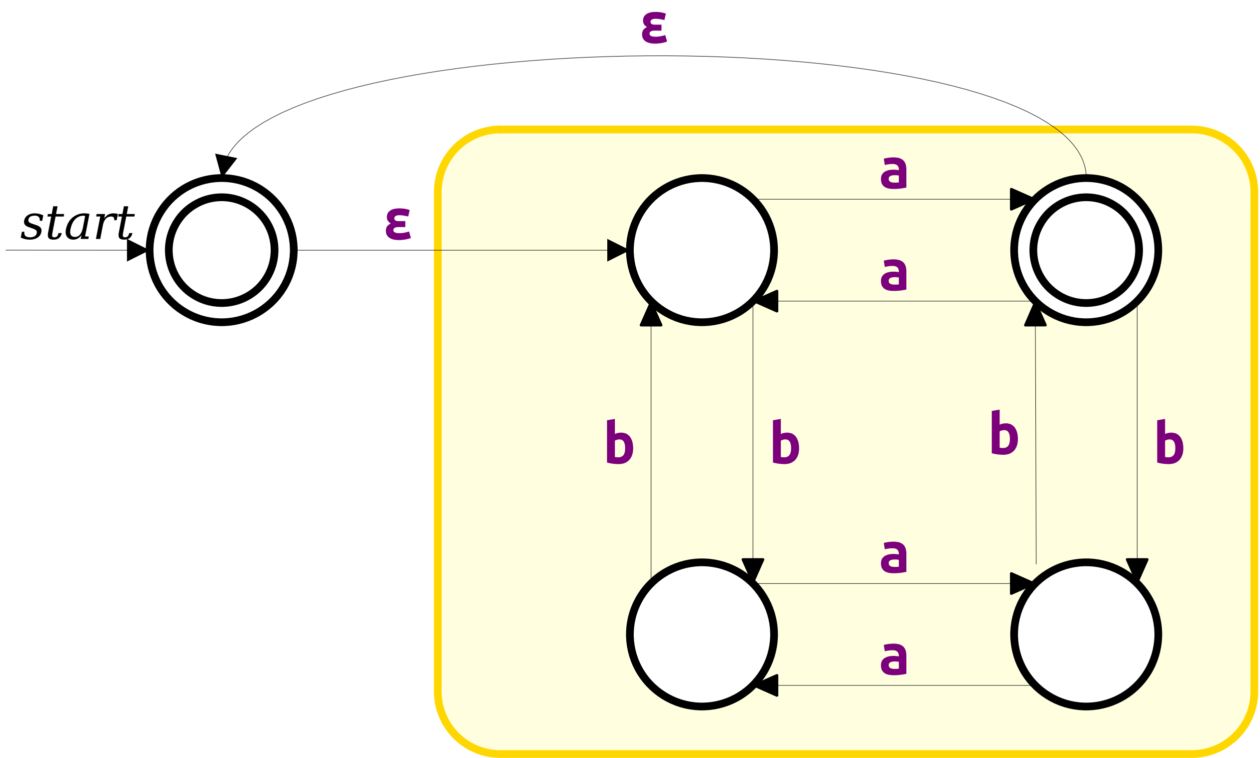
Construct an NFA for L_1L_2.



DFA for L

$L = \{ w \in \{\mathtt{a}, \mathtt{b}\}^* \mid w$ has an odd number of $\mathtt{a}$'s and an even number of $\mathtt{b}$'s $\}$

Construct an NFA for L^*.



DFA for L

$L = \{ w \in \{\mathtt{a}, \mathtt{b}\}^* \mid w$ has an odd number of $\mathtt{a}$'s and an even number of $\mathtt{b}$'s $\}$

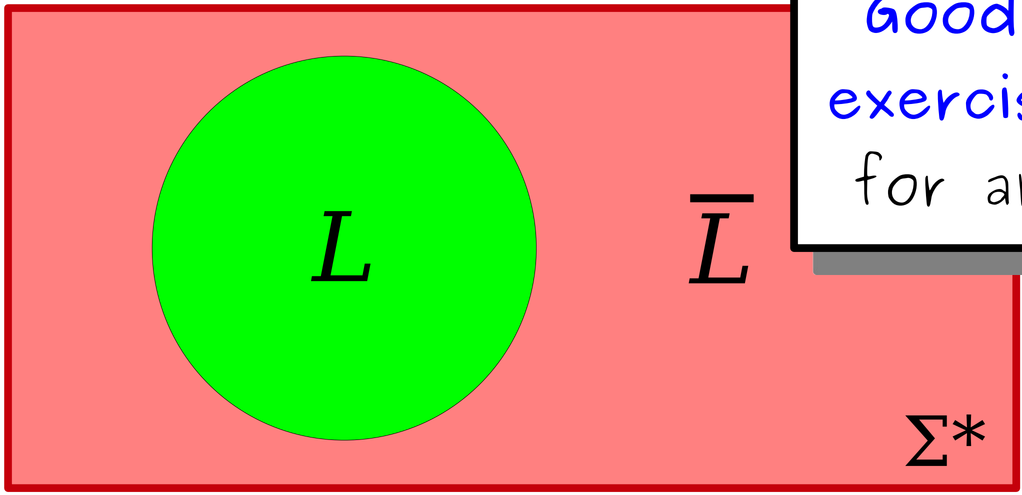
Construct an NFA for L^*.

Appendix 2: More Closure Properties

The Complement of a Language

- Given a language $L \subseteq \Sigma^*$, the ***complement*** of that language (denoted $\overline{L}$) is the language of all strings in Σ^* that aren't in L.
- Formally:

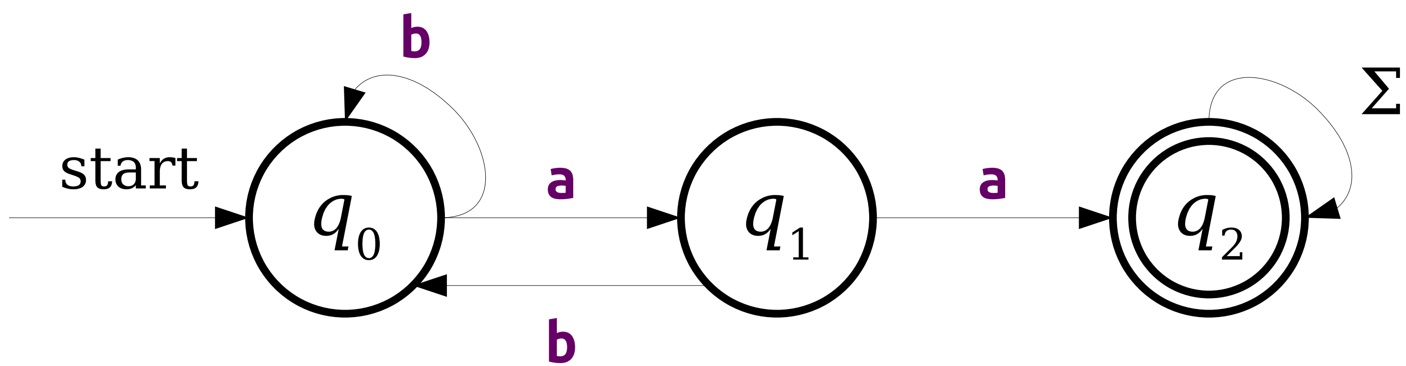
$$\overline{L} = \Sigma^* - L$$



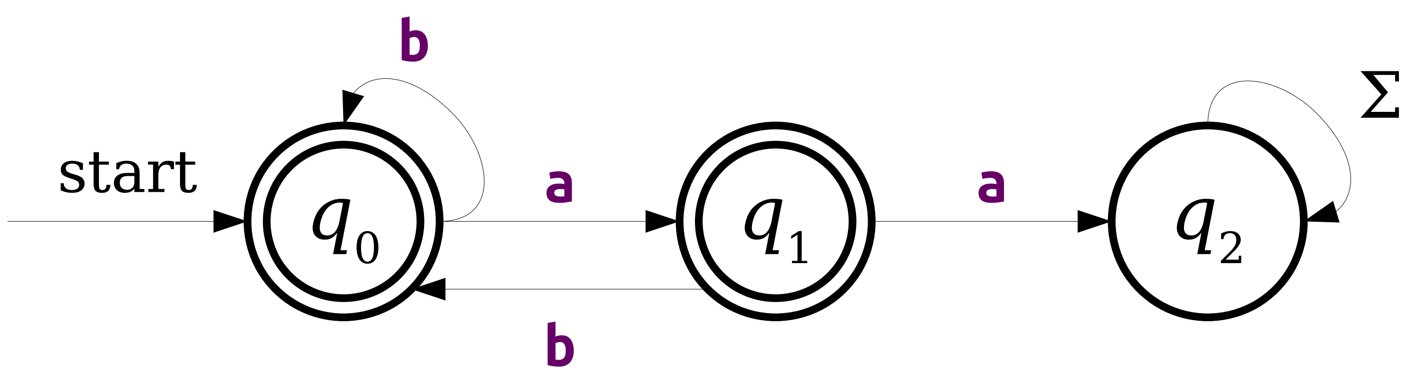
Good proofwriting exercise: prove $\overline{\overline{L}} = L$ for any language L.

Complementing Regular Languages

$L = \{ w \in \{\texttt{a}, \texttt{b}\}^* \mid w \text{ contains } \texttt{aa} \text{ as a substring } \}$

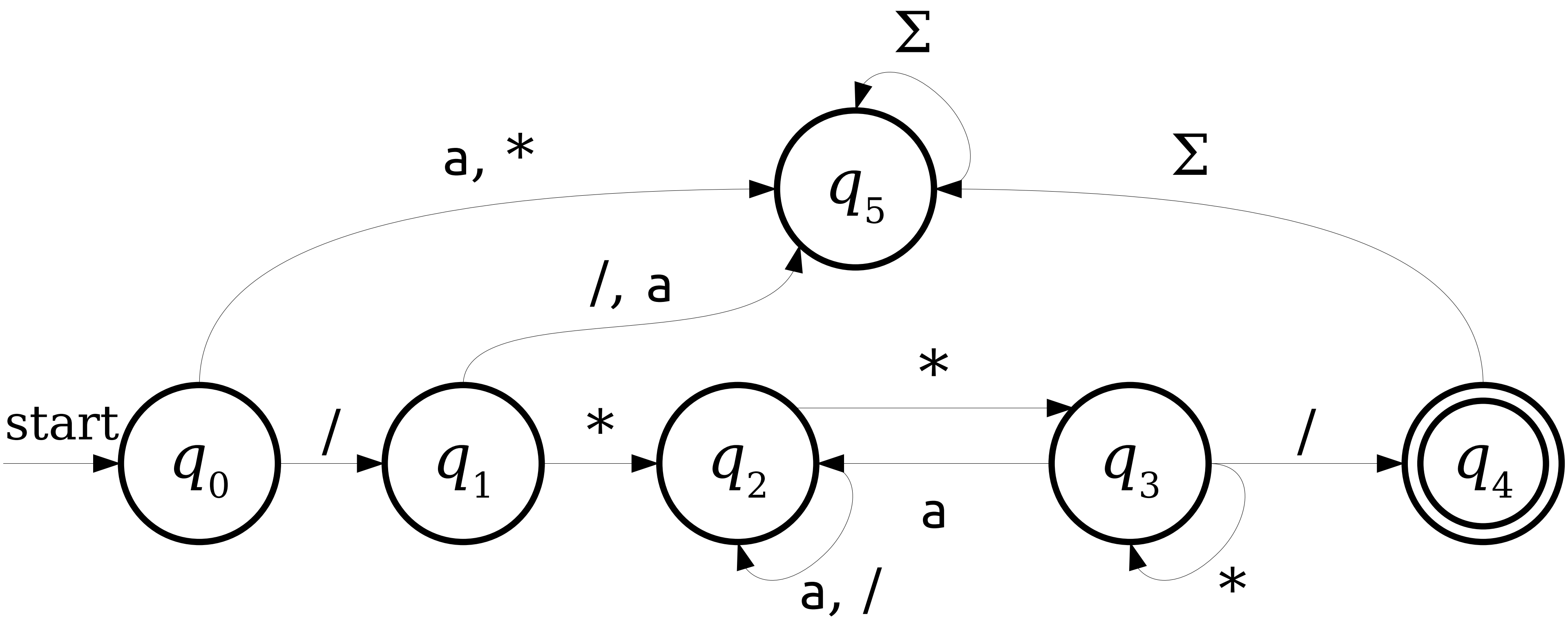


$\overline{L} = \{ w \in \{\texttt{a}, \texttt{b}\}^* \mid w \textit{ does not} \text{ contain } \texttt{aa} \text{ as a substring } \}$



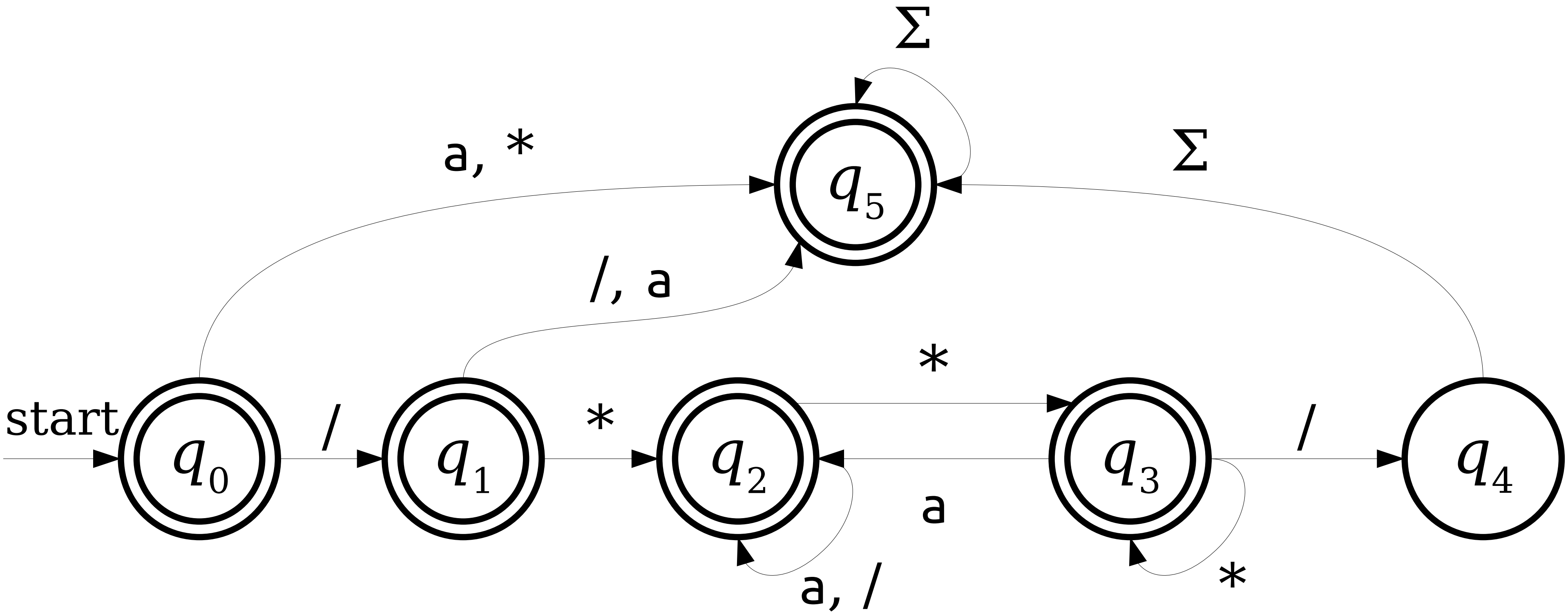
Complementing Regular Languages

$L = \{ \ w \in \{$**a**, *****, **/**$\}^* \mid w$ represents a C-style comment $\}$



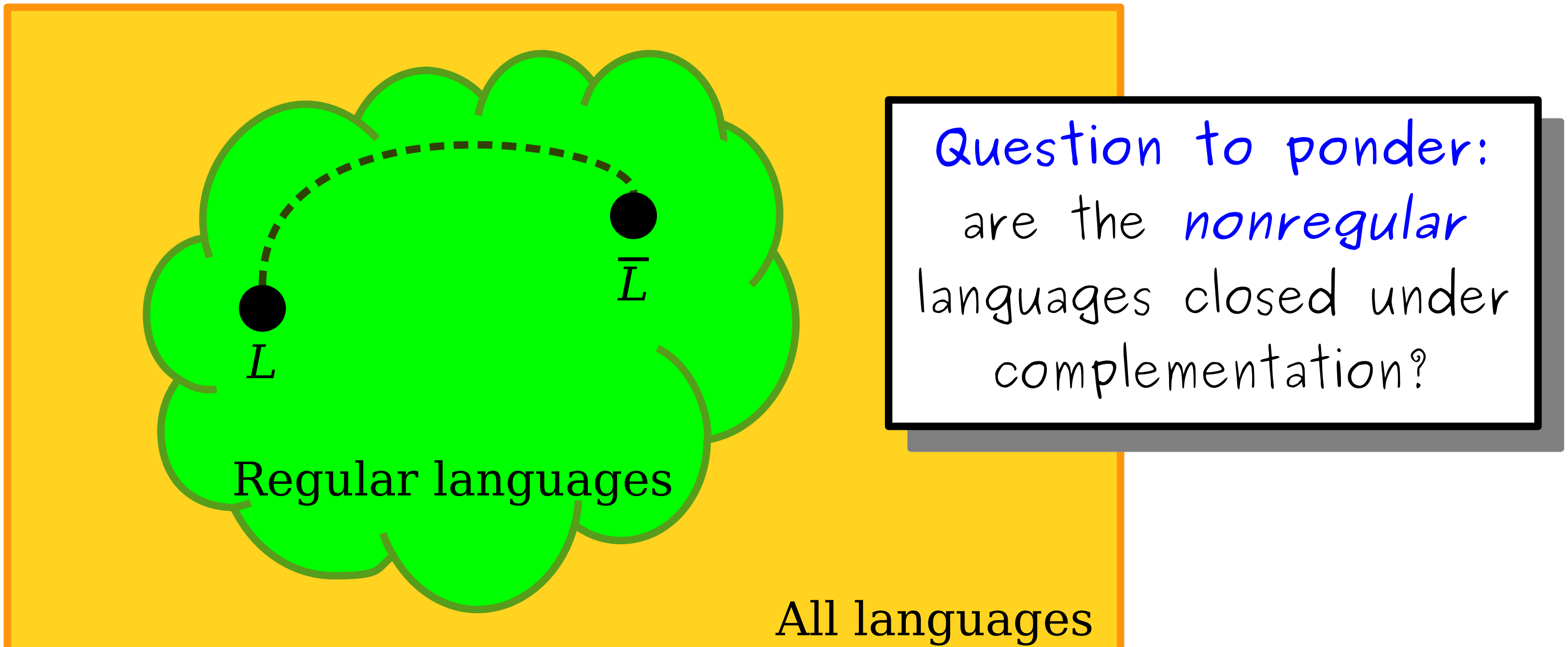
Complementing Regular Languages

$\overline{L}$ = { w ∈ {**a**, *****, **/**}* | w ***doesn't*** represent a C-style comment }



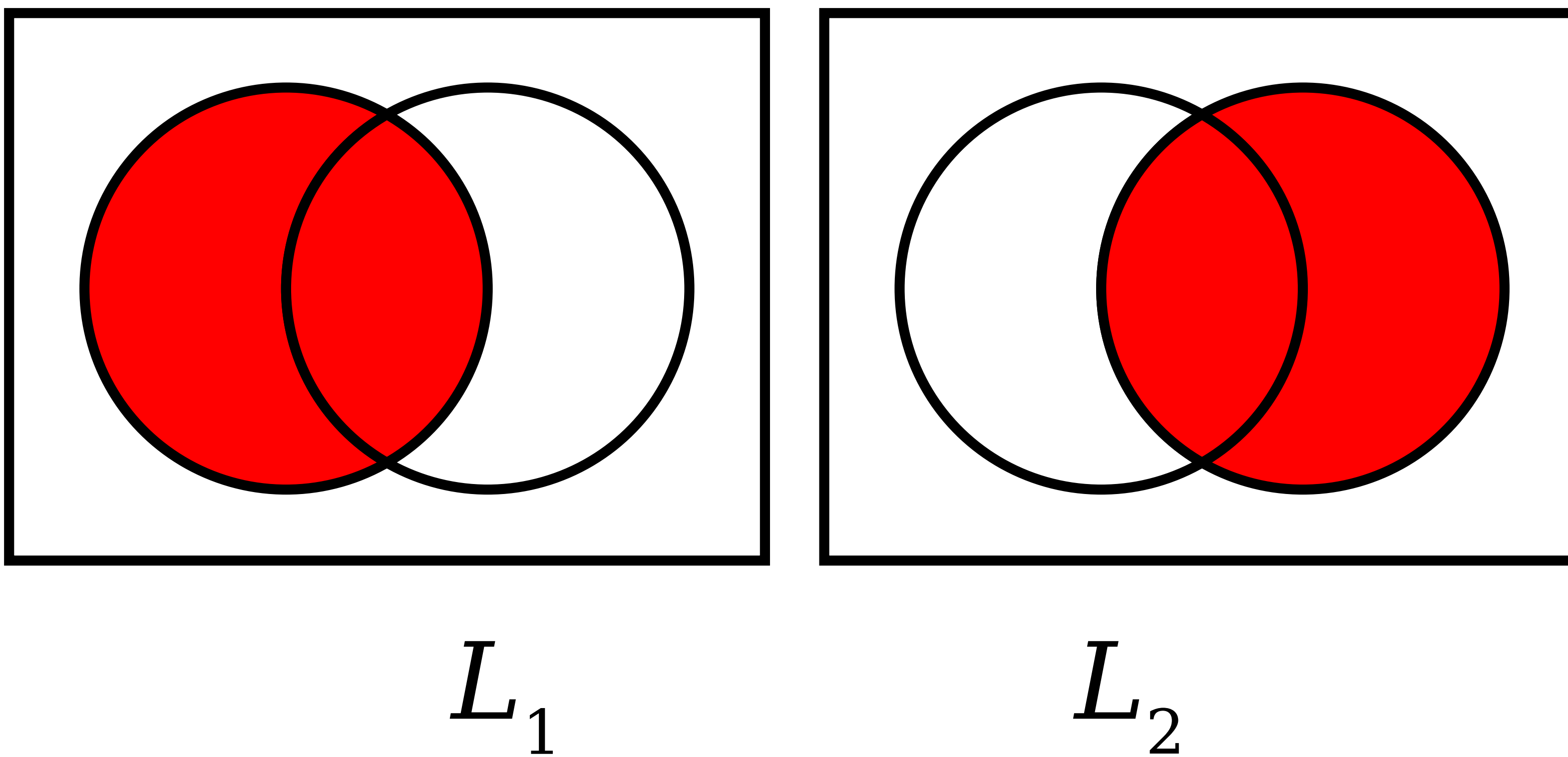
Closure Properties

- ***Theorem:*** If L is a regular language, then $\overline{L}$ is also a regular language.
- As a result, we say that the regular languages are ***closed under complementation***.



Closure Under Intersection

- If L_1 and L_2 are languages over Σ, then $L_1 \cap L_2$ is the language of strings in both L_1 and L_2.
- Question: If L_1 and L_2 are regular, is $L_1 \cap L_2$ regular as well?



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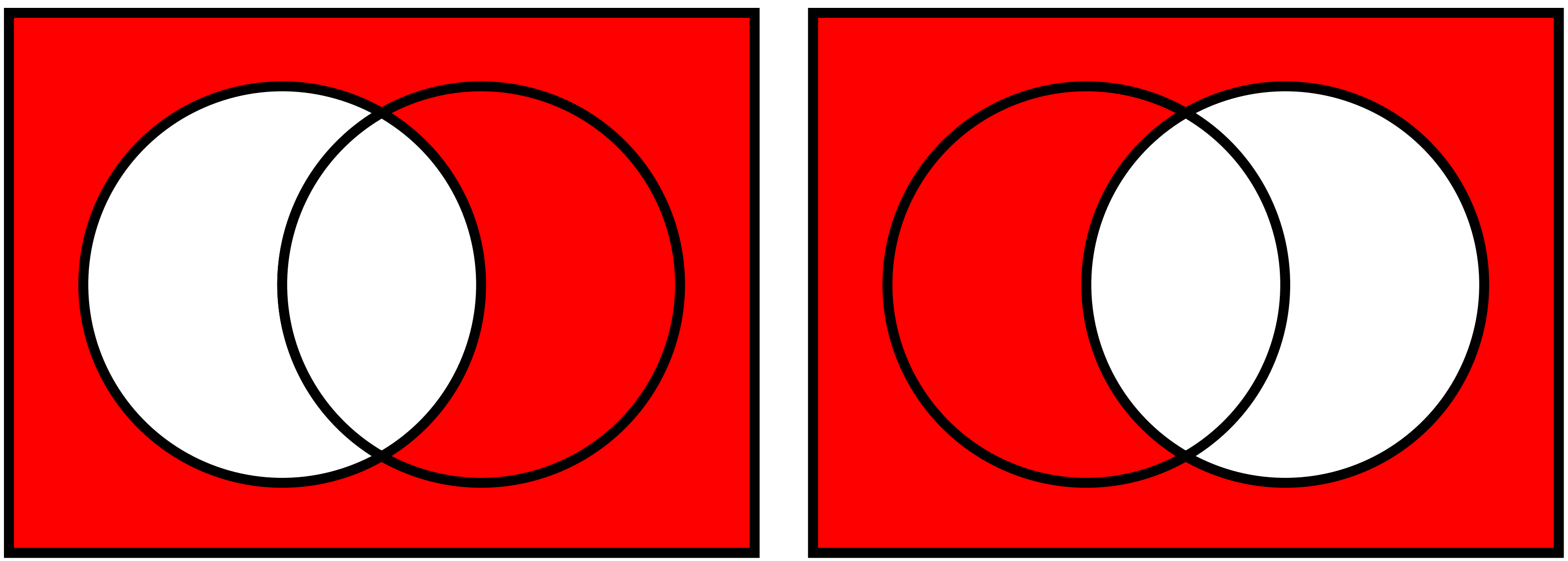

$\overline{L}_1$ $\overline{L}_2$

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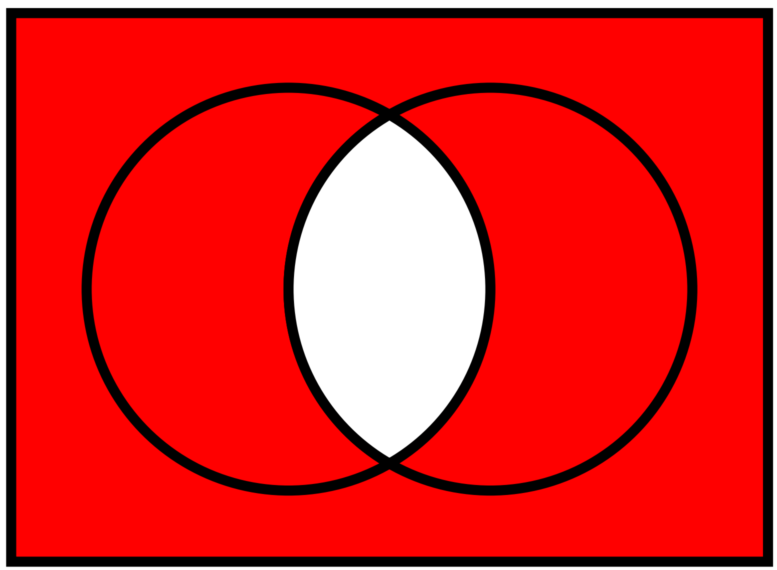

$$\overline{L}_1 \cup \overline{L}_2$$

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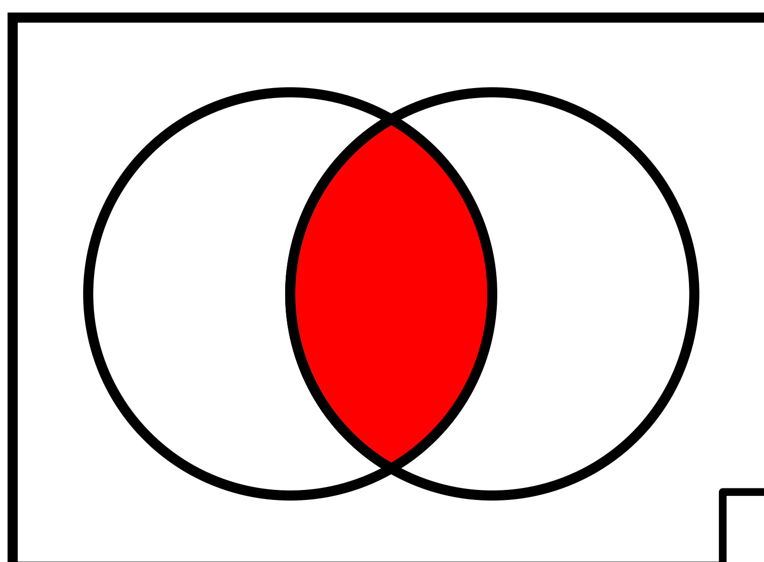

$$\overline{\overline{L_1} \cup \overline{L_2}}$$

Hey, it's De Morgan's laws!